

TRANSVERSE KÄHLER STRUCTURES ON CENTRAL FOLIATIONS OF COMPLEX MANIFOLDS

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ABSTRACT. For a compact complex manifold, we introduce holomorphic foliations associated with certain abelian subgroups of the automorphism group. Such foliations are generalizations of holomorphic principal torus bundles. If there exists a transverse Kähler structure on such a foliation, then we obtain a nice differential graded algebra which is quasi-isomorphic to the de Rham complex and a nice differential bigraded algebra which is quasi-isomorphic to the Dolbeault complex like the formality of compact Kähler manifolds. Moreover, under certain additional condition, we can develop Morgan's theory of mixed Hodge structures as similar to the study on smooth algebraic varieties.

1. INTRODUCTION

Let M be a manifold with an action of a connected Lie group H . Suppose that the H -action is *local free*, i.e. the isotropy group at any point is discrete. Denote by \mathfrak{h} the Lie algebra of H . Then the action induces an injective Lie algebra homomorphism $\mathfrak{h} \ni v \mapsto X_v \in \mathcal{C}^\infty(TM)$ and this gives an integrable distribution. Hence the local free action gives a foliation \mathcal{F} . If M is a complex manifold, H is a complex Lie group and the H -action is holomorphic, then the foliation \mathcal{F} is holomorphic. We are interested in transverse geometry on a foliated manifold (M, \mathcal{F}) . Denote by $\Omega^*(M)$ the space of the differential forms on M . We say that $\omega \in \Omega^*(M)$ is *basic* if $i_{X_v}\omega = 0$ and $L_{X_v}\omega = 0$ for any $v \in \mathfrak{h}$ where i_{X_v} and L_{X_v} are the inner product and the Lie derivation respectively. For a holomorphic foliation \mathcal{F} on a complex manifold M with the complex structure J , a transverse Kähler structure on \mathcal{F} is a closed real basic $(1, 1)$ -form ω so that $\omega(X, JX) \geq 0$ for any $X \in TX$ and the equality holds if and only if $X \in T\mathcal{F}$.

In this paper we consider the geometric structure which is a generalization of the structure of holomorphic principal torus bundle. Let M be a compact complex manifold. We consider canonical foliations associated with central subgroups of the automorphism of M . Let G_M be the identity component of the group of all biholomorphisms on M . G_M is a complex Lie group (see [3]). Let T be a maximal compact torus of G_M and \mathfrak{t} the Lie algebra of T . Let J be the complex structure on the Lie algebra of G_M . Put

$$\mathfrak{h}_M := \mathfrak{t} \cap J\mathfrak{t}$$

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and denote by H_M the corresponding Lie subgroup of G_M . Then H_M acts on M local freely (see [11]). Moreover we can say that H_M is a central subgroup in G_M and H_M does not depend on the choice of T (see Lemma 2.1). By the local freeness, for any connected subgroup $H \subset H_M$, we have the holomorphic foliation \mathcal{F}_H . We call \mathcal{F}_H a *central foliation* associated with H . In general, a connected complex abelian Lie group H is not a complex torus. If H is a compact complex torus, then the central foliation \mathcal{F}_H associated with H gives a holomorphic principal Seifert bundle structure on a complex manifold M over the complex orbifold M/H (see [20]). Moreover, if the action of H is free, then such Seifert bundle structure is a holomorphic principal torus bundle structure. Conversely, holomorphic principal torus bundle structure gives a holomorphic free complex torus action. Thus, we can say that a central foliation is a generalization of the structure of holomorphic principal torus bundle.

The purpose of this paper is to study (non-Kähler) complex manifolds admitting a transverse Kähler structure on a central foliation \mathcal{F}_H . Simple examples are holomorphic principal torus bundles over compact Kähler manifolds. It is known that any product $S^{2m-1} \times S^{2n-1}$ of odd dimensional spheres admits a complex structure so that such complex manifold admits a holomorphic principal torus bundle over $\mathbb{C}P^{m-1} \times \mathbb{C}P^{n-1}$ (Calabi-Eckmann [4]). Extending Calabi-Eckmann's construction, Meersseman constructed a large class of compact complex manifolds. Such complex manifolds are called *LVM-Manifolds* (see [16], [17]). Every LVM-manifold admits a transverse Kähler structure on a central foliation \mathcal{F}_H (see [11]). Moreover, there exists a LVM-manifold whose transverse Kähler structure on a central foliation does not gives a holomorphic principal Seifert bundle structure.

For a complex manifold M admitting a holomorphic foliation \mathcal{F} , we consider the basic de Rham complex $\Omega_B^*(M)$, basic Dolbeault complex $\Omega_B^{*,*}(M)$, basic de Rham cohomology $H_B^*(M)$ and basic Dolbeault cohomology $H_B^{*,*}(M)$ for \mathcal{F} . It is known that if there exists a transverse Kähler structure on \mathcal{F} , there is the Hodge decomposition

$$H_B^r(M, \mathbb{C}) = \oplus_{p+q=r} H_B^{p,q}(M),$$

$$\overline{H_B^{p,q}(M)} = H_B^{q,p}(M)$$

(see [6]).

Definition 1.1. • For a manifold M , a (de Rham) model of M is a differential graded algebra (shortly DGA) A^* such that A^* is quasi-isomorphic to the de Rham complex $\Omega^*(M)$ i.e. there exists a sequence of DGA homomorphisms

$$A^* \leftarrow C_1^* \rightarrow C_2^* \leftarrow \cdots \leftarrow C_n^* \rightarrow \Omega^*(M)$$

such that all the morphisms are quasi-isomorphisms (i.e. inducing cohomology isomorphisms).

• For a complex manifold M , a Dolbeault-model of M is a differential bi-graded algebra (shortly DBA) $B^{*,*}$ such that $B^{*,*}$ is quasi-isomorphic to the Dolbeault complex $\Omega^{*,*}(M)$ i.e. there exists a sequence of DBA homomorphisms

$$B^{*,*} \leftarrow C_1^{*,*} \rightarrow C_2^{*,*} \leftarrow \cdots \leftarrow C_n^{*,*} \rightarrow \Omega^{*,*}(M)$$

such that all the morphisms are quasi-isomorphisms.

On a compact Kähler manifold M , it is known that the de Rham cohomology $H^*(M)$ with the trivial differential is a model of M (Formality [5]) and the Dolbeault cohomology $H^{*,*}(M)$ with the trivial differential is a Dolbeault-model of M (Dolbeault-Formality [19]). In this paper we prove:

Theorem 1.2 (=Theorem 4.12). *Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a k -dimensional central foliation \mathcal{F}_H . Then we can obtain a model \mathcal{A}^* with a differential d and Dolbeault-model $\mathcal{B}^{*,*}$ with a differential $\bar{\partial}$ satisfying the following conditions:*

- (1) $\mathcal{A}^* = H_B^*(M) \otimes \bigwedge W$ as a graded algebra and $\mathcal{B}^{*,*} = H_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1})$ as a bi-graded algebra.
- (2) W is a $2k$ -dimensional graded vector space of degree 1 and $W^{1,0}$ (resp. $W^{0,1}$) is a k -dimensional bi-graded vector space of degree $(1,0)$ (resp. $(0,1)$) so that we have a decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$.
- (3) The differentials d and $\bar{\partial}$ are trivial on $H_B^*(M)$ and $H_B^{*,*}(M)$ respectively.
- (4) $dW \subset H_B^2(M)$, $\bar{\partial}W^{1,0} \subset H_B^{1,1}(M)$ and $\bar{\partial}W^{0,1} \subset H_B^{0,2}(M)$.

If \mathcal{F}_H gives a holomorphic principal torus bundle structure, then the basic cohomologies $H_B^*(M)$ and $H_B^{*,*}(M)$ are identified with the cohomologies of the base space. In [23], Tanré constructed Dolbeault Models of holomorphic principal torus bundles over compact Kähler manifolds.

We will explain the constructions of \mathcal{A}^* and $\mathcal{B}^{*,*}$ in more explicitly. A vector space W as in the theorem is a finite dimensional subspace $W \subset \Omega^1(M)^H$ so that:

- $dW \subset \Omega_B^2(M)$
- the bilinear map $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v} w \in \mathbb{R}$ is non-degenerate

where \mathfrak{h} is the Lie algebra of H and $\Omega^*(M)^H$ is the space of the H -invariant differential forms. By the complex structure of \mathfrak{h} , we take the decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$. The differential $W \rightarrow H_B^2(M)$ (resp. $W^{1,0} \rightarrow H_B^{1,1}(M)$ and $W^{0,1} \rightarrow H_B^{0,2}(M)$) for $\mathcal{A}^* = H_B^*(M) \otimes \bigwedge W$ (resp. $\mathcal{B}^{*,*} = H_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1})$) is given by $W \ni w \mapsto [dw]_B \in H_B^2(M)$ (resp. $W^{1,0} \ni w \mapsto [\bar{\partial}w]_B \in H_B^{1,1}(M)$ and $W^{0,1} \ni w \mapsto [\bar{\partial}w]_B \in H_B^{0,2}(M)$).

Definition 1.3. A transverse Kähler structure on \mathcal{F}_H is *special* if we can take W as above so that the image of the map $W \ni w \mapsto [dw]_B \in H_B^2(M)$ is contained in $H_B^{1,1}(M)$ for the Hodge structure

$$H_B^2(M) = H_B^{2,0}(M) \oplus H_B^{1,1}(M) \oplus H_B^{0,2}(M).$$

Remark 1.4. The Hodge decomposition does not depend on a transverse Kähler structure. Hence, in fact, this speciality is determined by a complex structure and a central foliation \mathcal{F}_H .

We prove:

Theorem 1.5 (=Theorem 5.10). *Let M be a compact complex manifold. We assume that M admits a special transverse Kähler structure on a k -dimensional central foliation \mathcal{F}_H . Then the de Rham cohomology of M admits an \mathbb{R} -mixed Hodge structure so that:*

- (1) $H^1(M, \mathbb{C}) = H_{1,0}^1 \oplus H_{0,1}^1 \oplus H_{1,1}^1$
- (2) $H^2(M, \mathbb{C}) = H_{2,0}^2 \oplus H_{1,1}^2 \oplus H_{0,2}^2 \oplus H_{2,1}^2 \oplus H_{1,2}^2 \oplus H_{2,2}^2$

and Sullivan's minimal model of the complex valued de Rham complex admits the Morgan's bigrading ([18]).

As in [18, Section 9, 10], we can study certain properties on the real homotopy type of a compact complex manifold M with a special transverse Kähler structure on a central foliation like smooth algebraic varieties. In particular, we can say "Morgan's test" i.e. the Malcev Lie algebra of $\pi_1(M)$ is the quotient of a free Lie algebra by an ideal generated by elements of degree $2 \leq d \leq 4$. Hence not every finitely generated group can be the fundamental group of a compact complex manifold with special transverse Kähler structure on a central foliation.

2. CENTRAL FOLIATIONS

Let M be a compact complex manifold. We consider canonical foliations associated with central subgroups of the automorphism of M . Let G_M be the identity component of the group of all biholomorphisms on M . G_M is a complex Lie group (see [3]). Denote by \mathfrak{g}_M the Lie algebra of G_M and by J the complex structure on \mathfrak{g}_M . Let T be a maximal compact torus of G_M and \mathfrak{t} the Lie algebra of T . Put

$$\mathfrak{h}_M := \mathfrak{t} \cap J\mathfrak{t}$$

and denote by H_M the corresponding Lie subgroup of G_M . Then H_M acts on M local freely (see [11]).

Lemma 2.1. *The following holds:*

- (1) *Elements in \mathfrak{h}_M centralize \mathfrak{g}_M .*
- (2) *\mathfrak{h}_M does not depend on the choice of T .*

Proof. Since T is compact, \mathfrak{g}_M is a unitary representation of T . In particular, \mathfrak{g}_M is a unitary representation of H_M . However, H_M is a holomorphic subgroup of G_M and hence \mathfrak{g}_M is a holomorphic representation of H_M . Therefore \mathfrak{g}_M is a trivial representation of H_M , showing Part (1).

Let T' be another maximal compact torus of G_M . Then, there exists $g \in G_M$ such that $gTg^{-1} = T'$ (see [10, Chapter XV, Section 3] for detail). Put

$$\mathfrak{h}' := \mathfrak{t}' \cap J\mathfrak{t}'.$$

Then, it follows from $gTg^{-1} = T'$ that $\text{Ad}_g(\mathfrak{h}_M) = \mathfrak{h}'$. On the other hand, by (1), we have Ad_g is the identity on \mathfrak{h}_M . Therefore \mathfrak{h}_M does not depend on the choice of T , proving (2). \square

We remark that any \mathbb{C} -subspace \mathfrak{h} of \mathfrak{h}_M defines a central foliation \mathcal{F}_H on M .

3. HIRSCH EXTENSIONS AND MINIMAL MODELS

In this section, DGAs are defined over $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or \mathbb{C} , if we do not specify.

Let (A^*, d_A) be a DGA. Let V be a vector space of degree k . For a linear map $\beta : V \rightarrow A^{k+1}$ with $d_A \circ \beta = 0$, we define a Hirsch extension (B, d_B) of A^* in degree k such that $B^* = A^* \otimes \bigwedge V$ with $\deg(v) = k$ for any $v \in V$, $d_B = d_A$ on A^* and $d_B = \beta$ on V . Defining the filtration on B^* by $F^p(B^*) = A^{\geq p} \otimes \bigwedge V$, we have the spectral sequence $E^{*,*}$ with $E_2^{p,q} = H^p(A) \otimes \bigwedge^q V$. Consider the composition $q \circ \beta : V \rightarrow H^{k+1}(A^*)$ where $q : \ker d_A \rightarrow H^*(A^*)$ is the quotient. The DGA structure of B^* is determined by the map $q \circ \beta$ (independent of a choice of β) ([8, 10.2]).

Lemma 3.1. *Let (A_1^*, d_{A_1}) and (A_2^*, d_{A_2}) be DGAs and $f : A_1^* \rightarrow A_2^*$ a quasi-isomorphism. Then for a Hirsch extension $A_1^* \otimes V$ (resp. $A_2^* \otimes V$), we have a Hirsch extension $A_2^* \otimes V$ (resp. $A_1^* \otimes V$) and quasi-isomorphism*

$$A_1^* \otimes V \rightarrow A_2^* \otimes V.$$

Proof. In case $A_1^* \otimes V$ ($\beta_1 : V \rightarrow A_1^{k+1}$) is given. Consider the Hirsch extension $A_2^* \otimes V$ given by $\beta_2 = f \circ \beta_1 : V \rightarrow A_2^*$ and the homomorphism $f \times id : A_1^* \otimes V \rightarrow A_2^* \otimes V$. Then we can easily show that $f \otimes id$ induces an isomorphism on the E_2 -term of the spectral sequence. Hence we can say $f \otimes id$ is a quasi-isomorphism.

In case $A_2^* \otimes V$ ($\beta_2 : V \rightarrow A_2^{k+1}$) is given. Since f is a quasi-isomorphism, we can take a linear map $\beta_1 : V \rightarrow A_1^*$ so that $d \circ \beta_1 = 0$ and $q \circ f \circ \beta_1 = q \circ \beta_2$. By the above argument, the Hirsch extension of A_2^* given by $\beta_2 : V \rightarrow A_1^{k+1}$ is identified with the one given by $f \circ \beta_1 : V \rightarrow A_1^{k+1}$. Under this identification, we have the homomorphism $f \otimes id : A_1^* \otimes V \rightarrow A_2^* \otimes V$ and as in the first case we can show that this homomorphism is a quasi-isomorphism. \square

Definition 3.2. A DGA \mathcal{M}^* is minimal if:

- $\mathcal{M}^0 = \mathbb{K}$.
 - $\mathcal{M}^* = \bigcup \mathcal{M}_i^*$ for a sequence of sub-DGAs
- $$k = \mathcal{M}_0^* \subset \mathcal{M}_1^* \subset \dots$$
- such that \mathcal{M}_{i+1}^* is a Hirsch extension of \mathcal{M}_i^* .
- $d\mathcal{M}^* \subset \mathcal{M}^+ \cdot \mathcal{M}^+$ where $\mathcal{M}^+ = \bigoplus_{j>0} \mathcal{M}^j$.

We say that a DGA \mathcal{M}^* is k -minimal if \mathcal{M}^* is minimal and $\bigoplus_{j>k} \mathcal{M}^j \subset \mathcal{M}^+ \cdot \mathcal{M}^+$. Equivalently each extension in a sequence for \mathcal{M}^* has degree at most k .

Definition 3.3. Let A^* be a DGA with $H^0(A^*) = \mathbb{K}$.

- A minimal DGA \mathcal{M}^* is the *minimal model* of A^* if there is a quasi-isomorphism $\mathcal{M} \rightarrow A^*$.
- A k -minimal DGA \mathcal{M}^* is the *k -minimal model* of A^* if there is a k -quasi-isomorphism $\mathcal{M}^* \rightarrow A^*$ i.e. homomorphism which induces an isomorphism $H^j(\mathcal{M}^*) \cong H^j(A^*)$ for $j \leq k$ and an injection $H^{k+1}(\mathcal{M}^*) \hookrightarrow H^{k+1}(A^*)$.

Theorem 3.4 ([22]). *For a DGA A^* with $H^0(A^*) = \mathbb{K}$, the minimal model and the k -minimal model exist and each of them is unique up to DGA isomorphism.*

The minimal models give the following “de Rham homotopy theorem”. We write the statement without details. See [22], [5], [8], [18] for the details.

Theorem 3.5. *Let M be a compact smooth manifold. Consider the de Rham complex $\Omega^*(M)$ as a DGA. Then*

- *The 1-minimal model of $\Omega^*(M)$ is the dual to the Lie algebra of the nilpotent completion of $\pi_1 M$*
- *If M is simply connected, then the minimal model of $\Omega^*(M)$ determines the real homotopy type of M .*

4. MODELS FOR TRANSVERSE KÄHLER TORUS ACTIONS

4.1. Models for compact Lie group actions. The following result is well-known (see e.g. [7]).

Proposition 4.1. *Let M be a connected compact manifold and K a connected compact Lie group. Assume that K acts on M . Then the inclusion*

$$\Omega^*(M)^K \subset \Omega(M)$$

induces a cohomology isomorphism.

Let M be a complex manifold. We consider the double complex $(\Omega^{*,*}(M), \partial, \bar{\partial})$. Suppose that a group K acts on M as biholomorphisms. Then the space $\Omega^{*,*}(M)^K$ of K -invariant differential forms is a sub-double complex.

Proposition 4.2. *Let M be a compact complex manifold and K a connected compact Lie group. Assume that K acts on M as biholomorphisms and the induced action on the Dolbeault cohomology is trivial. Then the inclusion*

$$\Omega^{*,*}(M)^K \subset \Omega^{*,*}(M)$$

induces an isomorphism on Dolbeault cohomology.

Proof. Take the normalized Haar measure $d\mu$ of K . Define

$$I : \Omega^{*,*}(M) \ni \omega \mapsto \int_{g \in K} g^* \omega d\mu \in \Omega^{*,*}(M)^K.$$

Then I commutes with Dolbeault operator $\bar{\partial}$. Hence we can easily see that the inclusion

$$\Omega^{*,*}(M)^K \subset \Omega^{*,*}(M)$$

induces an injection on Dolbeault cohomology.

Since the induced action on the Dolbeault cohomology is trivial, for a $\bar{\partial}$ -closed form $\omega \in \Omega^{*,*}(M)$ and any $g \in G$, we have $\theta_g \in \Omega^{*,*}(M)$ such that

$$\omega - g^* \omega = \bar{\partial} \theta_g.$$

By using Green operator, we can take θ_g smooth on G . Integrating by $d\mu$, we have

$$\omega - I\omega = \bar{\partial} \int_{g \in K} \theta_g d\mu.$$

Hence the inclusion

$$\Omega^{*,*}(M)^K \subset \Omega^{*,*}(M)$$

induces a surjection on Dolbeault cohomology. □

Corollary 4.3. *Let M be a compact complex manifold and K a connected compact Lie group acting on M as biholomorphisms. Let H be a dense Lie subgroup of K such that H is a complex Lie group and the restricted action of K to H on M is holomorphic. Then, the inclusion $\Omega^{*,*}(M)^K \subset \Omega(M)$ induces an isomorphism on Dolbeault cohomology.*

Proof. By Proposition 4.2, we only need to know that the representation of K on $H^{*,*}(M)$ is trivial under the assumptions of this proposition. Since K acts on M as biholomorphisms, the representation of K on $H^{*,*}(M)$ is \mathbb{C} -linear. Since K is compact, there exists a Hermitian inner product on $H^{*,*}(M)$ that is invariant under K .

Consider the induced representation $H \rightarrow GL(H^{*,*}(M))$. Since H is a complex Lie group and the restricted action of K to H on M is holomorphic, this representation is holomorphic ([14]). On the other hand, by the above argument, this

representation is unitary. Therefore the representation of H on $H^{*,*}(M)$ is trivial. Since H is dense in K , the representation of K on $H^{*,*}(M)$ is also trivial. The proposition is proved. \square

4.2. Models for torus actions. Let T be a compact torus and H a connected Lie subgroup (not necessary to be closed in T). Let M be a paracompact smooth manifold equipped with an action of T . In this section, we suppose that the restricted action of T to H on M is local free. Denote by \mathfrak{t} and \mathfrak{h} the Lie algebras of T and H respectively.

Lemma 4.4. *There exists a \mathfrak{h} -valued 1-form ω on M such that*

- (1) $i_{X_v}\omega = v$ for all $v \in \mathfrak{h}$,
- (2) ω is T -invariant.

Proof. Since T is compact and M is paracompact, it follows from the slice theorem that there exists a locally finite open covering $\mathcal{U} = \{U_\lambda\}_\lambda$ such that each U_λ is T -equivariantly diffeomorphic to $T \times_{T_\lambda} V_\lambda$ via φ_λ , where T_λ is a closed subgroup of T and V_λ is a representation space of T_λ . Let $\pi : T \times_{T_\lambda} V_\lambda \rightarrow T/T_\lambda$ be the map induced by the first projection $T \times V_\lambda \rightarrow T$. Since the action of H on M is local free, we have that $\mathfrak{h} \cap \mathfrak{t}_\lambda = 0$. Therefore there exists a \mathfrak{h} -valued 1-form ω_λ on T/T_λ that satisfies the conditions (1) and (2). Since π and φ_λ are T -invariant, the pull-back $(\pi \circ \varphi_\lambda)^*\omega_\lambda$ that is a \mathfrak{h} -valued 1-form on U_λ also satisfies the conditions (1) and (2).

Let $\{\rho_\lambda\}$ be a partition of unity subordinate to the open covering \mathcal{U} . Averaging ρ_λ with the normalized Haar measure on T , we may assume that every ρ_λ is T -invariant. Then the 1-form

$$\omega := \sum_\lambda \rho_\lambda (\pi \circ \varphi_\lambda)^* \omega_\lambda$$

on M satisfies the condition (1) and (2), as required. \square

Since the H -action is local free, the H -action implies the foliation \mathcal{F} on M . Denote by T' the closure of H .

Lemma 4.5.

$$\Omega^*(M)^{T'} = \Omega^*(M)^H.$$

Proof. Since $H \subset T'$, we have the inclusion $\Omega^*(M)^{T'} \subset \Omega^*(M)^H$. For $g \in T'$, take a sequence $\{g_i\}_{i=1, \dots}$ of elements in H so that $\lim_{i \rightarrow \infty} g_i = g$. Then we have

$$g^*\omega = \lim_{i \rightarrow \infty} g_i^*\omega = \omega$$

for any $\omega \in \Omega^*(M)^H$, showing the opposite inclusion $\Omega^*(M)^{T'} \supset \Omega^*(M)^H$. The lemma is proved. \square

Consider the basic forms

$$\Omega_B^*(M) = \{\omega \in \Omega^*(M) \mid i_{X_v}\omega = L_{X_v}\omega = 0, \forall v \in \mathfrak{h}\}.$$

Suppose that we have a finite dimensional subspace $W \subset \Omega^1(M)^H$ so that:

- $dW \subset \Omega_B^2(M)$
- the bilinear map $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v}w \in \mathbb{R}$ is non-degenerate

We can always obtain such W . Take a \mathfrak{h} -valued 1-form ω as in Lemma 4.4. For a basis v_1, \dots, v_k of \mathfrak{h} , we may write $w = \sum w_i v_i$ with 1-forms w_1, \dots, w_k . We claim that $dw_i \in \Omega_B^2(M)$. Since w_i is T -invariant, by Cartan formula we have

$$0 = L_{X_v} w_i = di_{X_v} w_i + i_{X_v} dw_i = i_{X_v} dw_i$$

for $v \in \mathfrak{h}$ because $i_{X_v} w_i$ is constant on M . By Cartan formula again,

$$L_{X_v} dw_i = di_{X_v} dw_i + i_{X_v} ddw_i = di_{X_v} dw_i.$$

This together with $i_{X_v} dw_i = 0$ yields that $dw_i \in \Omega_B^*(M)$. Then $W = \langle w_1, \dots, w_l \rangle$ is a desired space.

Proposition 4.6. *We have the decomposition*

$$\Omega^*(M)^H = \Omega_B^*(M) \otimes \bigwedge W.$$

Proof. For $\omega \in \Omega_B^*(M)$, the condition $L_{X_v} \omega = 0, \forall v \in \mathfrak{h}$ implies $\omega \in \Omega^*(M)^H$. Since $W \subset \Omega^1(M)^H$ and $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v} w \in \mathbb{R}$ is non-degenerate, we have the inclusion

$$\Omega_B^*(M) \otimes \bigwedge W \subset \Omega^*(M)^H.$$

We will show that $\Omega^*(M)^H \subset \Omega_B^*(M) \otimes \bigwedge W$. We say that $\omega \in \Omega^*(M)^H$ is of q -type if for any $v_1, \dots, v_q \in \mathfrak{h}$ we have

$$i_{X_{v_1}} \dots i_{X_{v_q}} \omega = 0.$$

If $\omega \in \Omega^*(M)^H$ is of 1-type, $\omega \in \Omega_B^*(M)$. Suppose that $\omega \in \Omega^*(M)^H$ is of q -type for $q \geq 2$. Then for any $v_1, \dots, v_{q-1} \in \mathfrak{h}$, we have

$$i_{X_{v_1}} \dots i_{X_{v_{q-1}}} \omega \in \Omega_B^*(M)$$

since for any $v \in \mathfrak{h}$ we have

$$i_{X_v} i_{X_{v_1}} \dots i_{X_{v_{q-1}}} \omega = 0$$

and

$$L_{X_v} i_{X_{v_1}} \dots i_{X_{v_{q-1}}} \omega = i_{X_{v_1}} \dots i_{X_{v_{q-1}}} L_{X_v} \omega = 0.$$

Take a basis v_1, \dots, v_k of \mathfrak{h} and the dual basis w_1, \dots, w_k of W given by $W \subset \Omega^1(M)^H$ and $\mathfrak{h} \times W \ni (v, w) \mapsto i_{X_v} w \in \mathbb{R}$. Then for $\omega \in \Omega^*(M)^H$ of q -type, we can see that the form

$$\omega' = \omega - \sum_{i_1 < i_2 < \dots < i_{q-1}} (i_{X_{v_{i_1}}} \dots i_{X_{v_{i_{q-1}}}} \omega) \wedge w_{i_1} \wedge \dots \wedge w_{i_{q-1}}.$$

is of $(q-1)$ -type. Hence, by the induction on q , we can prove $\omega \in \Omega_B^*(M) \otimes \bigwedge W$ and so the proposition follows. \square

By Propositions 4.1, 4.6 and Lemma 4.5, we have the following result.

Corollary 4.7. *We have an injection*

$$\Omega_B^*(M) \otimes \bigwedge W \rightarrow \Omega^*(M)$$

which induces a cohomology isomorphism.

Proposition 4.8. *Suppose that $\dim M = n + k$. Then, $H^{n+k}(M) \cong H_B^n(M)$. In particular, \mathcal{F} is homologically oriented if M is compact and oriented.*

Proof. By Proposition 4.1 and Lemma 4.5, we can choose a representative α of an element in $H^{n+k}(M)$ so that α sits in $\Omega^{n+k}(M)^H$. By Proposition 4.6, there uniquely exists $\beta \in \Omega_B^n(M)$ such that $\alpha = \beta \wedge w_1 \wedge \cdots \wedge w_k$. Conversely, for $\beta \in \Omega_B^n(M)$, $\alpha := \beta \wedge w_1 \wedge \cdots \wedge w_k \in \Omega^n(M)^H$. Thanks to the degrees, α and β both are automatically closed. Therefore it suffices to show that α is exact if and only if β is exact (in the sense of basic).

Let $\alpha' \in \Omega^{n+k-1}(M)^H$ such that $d\alpha' = \alpha$. By Proposition 4.6, we can write

$$\alpha' = \beta' \wedge w_1 \wedge \cdots \wedge w_k + \sum_{i=1}^l \beta_i \wedge w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_k$$

with $\beta' \in \Omega_B^{n-1}(M)$ and $\beta_i \in \Omega_B^n(M)$ for $i = 1, \dots, l$. Then, it follows from $dw_j \in \Omega_B^2(M)$ that $\alpha = d\alpha' = d\beta' \wedge w_1 \wedge \cdots \wedge w_k$. In particular, $\beta = d\beta'$.

To see the converse, let $\beta' \in \Omega_B^{n-1}(M)$ such that $d\beta' = \beta$. Then

$$d(\beta' \wedge w_1 \wedge \cdots \wedge w_k) = \beta \wedge w_1 \wedge \cdots \wedge w_k + (-1)^{n-1} \beta' \wedge d(w_1 \wedge \cdots \wedge w_k).$$

Since $\beta' \wedge dw_j = 0$ by the degree, we have that

$$\alpha = d(\beta' \wedge w_1 \wedge \cdots \wedge w_k),$$

showing the equivalence of exactness between α and β . The proposition is proved. \square

4.3. Models for transverse Kähler torus actions. Let M be a compact complex manifold and T a compact torus acting on M as holomorphic transformations. Let H be a dense Lie subgroup of T such that H is a complex Lie group and the restricted action of T to H on M is holomorphic and local free. We take W as the last subsection. By $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{h}^{1,0} \oplus \mathfrak{h}^{0,1}$, we have $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$ and obtain the DBA

$$\Omega_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1})$$

with the Dolbeault operator $\bar{\partial}$.

By Proposition 4.2, 4.6 and Lemma 4.5, we have the following result.

Corollary 4.9. *We have an injection*

$$\Omega_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1}) \rightarrow \Omega^{*,*}(M)$$

which induces a cohomology isomorphism.

We consider the bi-graded bi-differential algebra (BBA) $(\Omega_B^{*,*}(M), \partial_B, \bar{\partial}_B)$. Denote $d^c = \sqrt{-1}(\bar{\partial}_B - \partial_B)$. Then d^c is a differential on $\Omega_B^*(M)$. We say that the $\partial_B \bar{\partial}_B$ -lemma holds if

$$\ker \partial_B \cap \ker \bar{\partial}_B \cap \text{im } d = \text{im } \partial_B \bar{\partial}_B.$$

If the $\partial_B \bar{\partial}_B$ -lemma holds, then we have the quasi-isomorphisms

$$\begin{aligned} (\ker d^c, d) &\rightarrow (\Omega_B^*(M), d), \\ (\ker d^c, d) &\rightarrow (H_B^*(M), 0), \\ (\ker \partial_B, \bar{\partial}_B) &\rightarrow (\Omega_B^{*,*}(M), \bar{\partial}_B) \end{aligned}$$

and

$$(\ker \partial_B, \bar{\partial}_B) \rightarrow (H_B^{*,*}(M), 0)$$

([5]).

Proposition 4.10. *Suppose that the $\partial_B \bar{\partial}_B$ -lemma holds. Then we have quasi-isomorphisms*

$$\begin{aligned} (\ker d^c \otimes \bigwedge W, d) &\rightarrow (\Omega_B^*(M) \otimes \bigwedge W, d), \\ (\ker d^c \otimes \bigwedge W, d) &\rightarrow (H_B^*(M) \otimes \bigwedge W, d), \\ (\ker \partial \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}) &\rightarrow (\Omega_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}) \end{aligned}$$

and

$$(\ker \partial_B \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}) \rightarrow (H_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1}), \bar{\partial}).$$

Proof. This follows from Lemma 3.1 immediately. \square

Theorem 4.11 ([6]+[5]). *Let M be a compact manifold with a homologically oriented $(H_B^{\text{codim } \mathcal{F}} \neq 0)$ transversely Kähler foliation \mathcal{F} . Then for the BBA $(\Omega_B^{*,*}(M), \partial_B, \bar{\partial}_B)$, the $\partial_B \bar{\partial}_B$ -lemma holds.*

This together with Proposition 4.8 implies the following result.

Theorem 4.12. *We assume that the foliation \mathcal{F} admits a transversely Kähler structure. Then the DGAs $\Omega^*(M)$ and $H_B^*(M) \otimes \bigwedge W$ (resp. DBAs $\Omega^{*,*}(M)$ and $H_B^{*,*}(M) \otimes \bigwedge (W^{1,0} \oplus W^{0,1})$) are quasi-isomorphic.*

5. MIXED HODGE STRUCTURES

5.1. Mixed Hodge structures.

Definition 5.1. An \mathbb{R} -Hodge structure of weight n on an \mathbb{R} -vector space V is a finite bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V_{p,q}$$

on the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that

$$\overline{V_{p,q}} = V_{q,p},$$

or equivalently, a finite decreasing filtration F^* on $V_{\mathbb{C}}$ such that

$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p}(V_{\mathbb{C}})}$$

for each p .

Definition 5.2. An \mathbb{R} -mixed-Hodge structure on an \mathbb{R} -vector space V is a pair (W_*, F^*) such that:

- (1) W_* is an increasing filtration which is bounded below,
- (2) F^* is a decreasing filtration on $V_{\mathbb{C}}$ such that the filtration on $Gr_n^W V_{\mathbb{C}}$ induced by F^* is an \mathbb{R} -Hodge structure of weight n .

We call W_* the weight filtration and F^* the Hodge filtration.

Proposition 5.3. ([18, Proposition 1.9]) *Let (W_*, F^*) be an \mathbb{R} -mixed-Hodge structure on an \mathbb{R} -vector space V . Define $V_{p,q} = R_{p,q} \cap L_{p,q}$ where $R_{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap F^p(V_{\mathbb{C}})$ and $L_{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap \overline{F^{q-i+1}(V_{\mathbb{C}})} + \sum_{i \geq 2} W_{p+q-i}(V_{\mathbb{C}}) \cap \overline{F^{q-i+1}(V_{\mathbb{C}})}$. Then we have the bigrading $V_{\mathbb{C}} = \bigoplus V_{p,q}$ such that: $W_i(V_{\mathbb{C}}) = \bigoplus_{p+q \leq i} V_{p,q}$ and $F^i(V_{\mathbb{C}}) = \bigoplus_{p \geq i} V_{p,q}$.*

Proposition 5.4. ([18, Proposition 1.11]) *Let V be an \mathbb{R} -vector space. We suppose that we have a bigrading $V_{\mathbb{C}} = \bigoplus V_{p,q}$ such that each $\left(\bigoplus_{p+q \leq i} V_{p,q}\right) \cap V$ is a real structure of $\bigoplus_{p+q \leq i} V_{p,q}$, $\overline{V_{p,q}} = V_{q,p}$ modulo $\bigoplus_{r+s < p+q} V_{r,s}$ and the grading $\text{Tot}^* V_{*,*}$ is bounded below where $\text{Tot}^r V_{*,*} = \bigoplus_{p+q=r} V_{p,q}$. Then the filtrations W and F such that $W_i(V) = \left(\bigoplus_{p+q \leq i} V_{p,q}\right) \cap V$ and $F^i(V_{\mathbb{C}}) = \bigoplus_{p \geq i} V_{p,q}$ give an \mathbb{R} -mixed Hodge structure on V .*

5.2. Morgan's Mixed Hodge diagrams.

Definition 5.5 ([18, Definition 3.5]). An \mathbb{R} -mixed-Hodge diagram is a pair of filtered \mathbb{R} -DGA (A^*, W_*) and bifiltered \mathbb{C} -DGA (E^*, W_*, F^*) and filtered DGA map $\phi : (A_{\mathbb{C}}^*, W_*) \rightarrow (E^*, W_*)$ such that:

- (1) ϕ induces an isomorphism $\phi^* : {}_W E_1^{*,*}(A_{\mathbb{C}}^*) \rightarrow {}_W E_1^{*,*}(E^*)$ where ${}_W E_*^{*,*}(\cdot)$ is the spectral sequence for the decreasing filtration $W^* = W_{-*}$.
- (2) The differential d_0 on ${}_W E_0^{*,*}(E^*)$ is strictly compatible with the filtration induced by F .
- (3) The filtration on ${}_W E_1^{p,q}(E^*)$ induced by F is an \mathbb{R} -Hodge structure of weight q on $\phi^*({}_W E_1^{*,*}(A^*))$.

Theorem 5.6 ([18, Theorem 4.3]). *Let $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$ be an \mathbb{R} -mixed-Hodge diagram. Define the filtration W'_* on $H^r(A^*)$ (resp. $H^r(E^*)$) as $W'_i H^r(A^*) = W_{i-r}(H^r(A^*))$ (resp. $W'_i H^r(E^*) = W_{i-r}(H^r(E^*))$). Then the filtrations W'_* and F^* on $H^r(E^*)$ give an \mathbb{R} -mixed-Hodge on $\phi^*(H^r(A^*))$.*

Theorem 5.7 ([18, Section 6, 8]). *Let $\{(A^*, W_*), (E^*, W_*, F^*), \phi\}$ be an \mathbb{R} -mixed-Hodge diagram. Then the minimal model (resp. 1-minimal model) \mathcal{M}^* of the DGA E^* with a quasi-isomorphism (resp. 1-quasi-isomorphism) $\phi : \mathcal{M}^* \rightarrow E^*$ satisfies the following conditions:*

- \mathcal{M}^* admits a bigrading

$$\mathcal{M}^* = \bigoplus_{p,q \geq 0} \mathcal{M}_{p,q}^*$$

such that $\mathcal{M}_{0,0}^ = \mathcal{M}^0 = \mathbb{C}$ and the product and the differential are of type $(0,0)$.*

- *For some real structure of \mathcal{M}^* , the bigrading $\bigoplus_{p,q \geq 0} \mathcal{M}_{p,q}^*$ induces an \mathbb{R} -mixed-Hodge structure.*
- *Consider the bigrading $H^*(E^*) = \bigoplus V_{p,q}$ for the \mathbb{R} -mixed-Hodge structure as in Theorem 5.6. Then $\phi^* : H^*(\mathcal{M}^*) \rightarrow H^*(E^*)$ sends $H^*(\mathcal{M}_{p,q}^*)$ to $V_{p,q}$.*

Proposition 5.8. *Let H^* be a graded commutative \mathbb{R} -algebra and V an \mathbb{R} -vector space with a linear map $\beta : V \rightarrow H^2$. We suppose the following conditions:*

- *For any p , H^p admits an \mathbb{R} -Hodge structure $H^p \otimes \mathbb{C} = \bigoplus_{s+t=p} H^{s,t}$ of weight p .*
- *For any p, q , the multiplication $H^p \times H^q \rightarrow H^{p+q}$ is a morphism of Hodge structure.*
- *V admits a Hodge structure $V \otimes \mathbb{C} = \bigoplus_{s+t=2} V^{s,t}$ of weight 2.*
- *$\beta : V \rightarrow H^2$ is a morphism of Hodge structure. (e.g. $\beta(V) \subset H^{1,1}$.)*

Regarding H^* as a DGA with trivial differential, we consider the Hirsch extension $A^* = H^* \otimes \bigwedge V$. Define the increasing filtration $W_* A^*$ as

$$W_k A^q = \oplus_{l \leq k} H^{q-l} \otimes \bigwedge^l V$$

and decreasing filtration $F^* A_{\mathbb{C}}^*$ as the Hodge filtration for the Hodge structure on $(H^p \otimes \mathbb{C}) \otimes \bigwedge^q (V \otimes \mathbb{C})$. Then the pair (A^*, W_*) , $(A_{\mathbb{C}}^*, W_*, F^*)$ and $\text{id} : A_{\mathbb{C}}^* \rightarrow A^*$ is an \mathbb{R} -mixed-Hodge diagram so that the cohomology $H^*(A_{\mathbb{C}}^*)$ decomposes as:

$$\begin{aligned} (1) \quad & H^1(A_{\mathbb{C}}^*) = H_{1,0}^1 \oplus H_{0,1}^1 \oplus H_{1,1}^1 \\ (2) \quad & H^2(A_{\mathbb{C}}^*) = H_{2,0}^2 \oplus H_{1,1}^2 \oplus H_{0,2}^2 \oplus H_{2,1}^2 \oplus H_{1,2}^2 \oplus H_{2,2}^2 \end{aligned}$$

where $H_{p,q}^* \subset H^*(A_{\mathbb{C}}^*)$ is the space of elements of bidegree (p, q) for the mixed structure.

Proof. ${}_W E_0^{-p,q}(A_{\mathbb{C}}^*) = (H^{q-2p} \otimes \mathbb{C}) \otimes \bigwedge^p (V \otimes \mathbb{C})$ and d_0 is trivial. ${}_W E_1^{-p,q}(A_{\mathbb{C}}^*) = (H^{q-2p} \otimes \mathbb{C}) \otimes \bigwedge^p (V \otimes \mathbb{C})$ and clearly F induces the Hodge structure of weight q .

We can easily check the assertions on bidegrees. \square

5.3. Mixed Hodge diagrams for transverse Kähler structures on central foliations. Let M be a compact complex manifold. We assume that M admits a transverse Kähler structure on a central foliation \mathcal{F}_H . We consider the BBA $(\Omega_B^{*,*}(M), \partial_B, \bar{\partial}_B)$. Define the basic Bott-Chern cohomology $H_{B,BC}^{*,*}(M)$ as

$$H_{B,BC}^{*,*}(M) = \frac{\text{Ker } \partial_B \cap \text{Ker } \bar{\partial}_B}{\text{Im } \partial_B \bar{\partial}_B}.$$

Then we have $\overline{H_{B,BC}^{p,q}(M)} = H_{B,BC}^{q,p}(M)$ and the natural algebra homomorphisms

$$\text{Tot}^* H_{B,BC}^{*,*}(M) \rightarrow H_B^*(M, \mathbb{C})$$

and

$$H_{B,BC}^{*,*}(M) \rightarrow H_B^{*,*}(M).$$

By $\partial_B \bar{\partial}_B$ -Lemma, we can say that these maps are isomorphisms (see [5, Remark 5.16]). Thus, we have the Hodge decomposition

$$H_B^r(M, \mathbb{C}) = \oplus_{p+q=r} H_B^{p,q}(M)$$

and

$$\overline{H_B^{p,q}(M)} = H_B^{q,p}(M).$$

We remark that this decomposition does not depend on the choice of a transverse Kähler structure.

Corollary 5.9. *Under the assumptions as in Theorem 4.12, for the model $\mathcal{A}^* = H_B^*(M) \otimes \bigwedge W$ as in Theorem 4.12, we can obtain the mixed Hodge diagram (\mathcal{A}^*, W_*) , $(\mathcal{A}_{\mathbb{C}}^*, W_*, F^*)$ as in Proposition 5.8.*

Theorem 5.10. *Let M be a compact complex manifold. We assume that M admits a special transverse Kähler structure on a central foliation \mathcal{F}_H . Consider the minimal model \mathcal{M} (resp. 1-minimal model) of $A_{\mathbb{C}}^*(M)$ with a quasi-isomorphism (resp. 1-quasi-isomorphism) $\phi : \mathcal{M} \rightarrow A_{\mathbb{C}}^*(M)$. Then we have:*

- (1) *The real de Rham cohomology $H^*(M, \mathbb{R})$ admits an \mathbb{R} -mixed-Hodge structure so that*
 - $H^1(M, \mathbb{C}) = H_{1,0}^1 \oplus H_{0,1}^1 \oplus H_{1,1}^1$
 - $H^2(M, \mathbb{C}) = H_{2,0}^2 \oplus H_{1,1}^2 \oplus H_{0,2}^2 \oplus H_{2,1}^2 \oplus H_{1,2}^2 \oplus H_{2,2}^2$

- (2) \mathcal{M}^* admits a bigrading

$$\mathcal{M}^* = \bigoplus_{p,q \geq 0} \mathcal{M}_{p,q}^*$$

such that $\mathcal{M}_{0,0}^* = \mathcal{M}^0 = \mathbb{C}$ and the product and the differential are of type $(0,0)$.

- (3) For some real structure of \mathcal{M}^* , the bigrading $\bigoplus_{p,q \geq 0} \mathcal{M}_{p,q}^*$ induces an \mathbb{R} -mixed-Hodge structure as Proposition 5.4.
- (4) Consider the bigrading $H^*(M, \mathbb{C}) = \bigoplus V_{p,q}$ for the \mathbb{R} -mixed-Hodge structure. Then the induced map $\phi^* : H^*(\mathcal{M}^*) \rightarrow H^*(M, \mathbb{C})$ sends $H^*(\mathcal{M}_{p,q}^*)$ to $V_{p,q}$.

6. EXAMPLES AND APPLICATIONS

6.1. Simple examples.

Example 6.1. Consider the product $S^{1,2n-1} = S^1 \times S^{2n-1}$ of a circle and a $(2n-1)$ -dimensional sphere equipped with a complex structure so that there exists a special transverse Kähler structure on a 1-dimensional central foliation \mathcal{F}_H . Then, by our results, $\Omega^*(S^{1,2n-1})$ is quasi-isomorphic to the DGA $\mathcal{A}^* = H_B^*(S^{1,2n-1}) \otimes \bigwedge W$. By $\dim H^1(S^{1,2n-1}) = 1$ and $H^1(S^{1,2n-1}, \mathbb{C}) = H_B^{1,0}(S^{1,2n-1}) \oplus H_B^{0,1}(S^{1,2n-1}) \oplus \ker d|_W$, we have $H_B^{1,0}(S^{1,2n-1}) \oplus H_B^{0,1}(S^{1,2n-1}) = 0$ and $\dim \ker d|_W = 1$. By $\dim H^2(S^{1,2n-1}) = 0$, the differential $d : W \rightarrow H_B^2(S^{1,2n-1})$ is surjective and hence $\dim H_B^2(S^{1,2n-1}) = 1$. Take $W = \langle x, y \rangle$ so that $dx \neq 0$ in $H_B^2(S^{1,2n-1})$ and $dy = 0$. We have $H_B^2(S^{1,2n-1}) = \langle dx \rangle$. Since $dx \in H_B^2(S^{1,2n-1})$ must contain transverse Kähler form, we have $(dx)^i \neq 0$ for any $i \leq n-1$. Inductively we can easily compute $H_B^{2i}(S^{1,2n-1}) = \langle (dx)^i \rangle$ and $H_B^{2i-1}(S^{1,2n-1}) = 0$ for $2 \leq i \leq n-1$.

Consider the Hodge decomposition

$$H_B^r(S^{1,2n-1}, \mathbb{C}) = \bigoplus_{p+q=r} H_B^{p,q}(S^{1,2n-1}).$$

Then we have $H_B^{i,i}(S^{1,2n-1}) = \langle (dx)^i \rangle$ for any $i \leq n-1$ and $H_B^{p,q}(S^{1,2n-1}) = 0$ for $p \neq q$. Take the decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$ with $W^{1,0} = \langle z \rangle$. Then we have $dz = cdx$ for some $c \in \mathbb{C}$. Thus we have $\bar{\partial}z = cdx$ and $\bar{\partial}\bar{z} = 0$. Hence $\Omega^{*,*}(S^{1,2n-1})$ is quasi-isomorphic to the DBA

$$B^{*,*} = \langle 1, dx, \dots, (dx)^{n-1} \rangle \otimes \bigwedge \langle z, \bar{z} \rangle.$$

Thus every complex structure on $S^{1,2n-1}$ with a special transverse Kähler structure on a 1-dimensional central foliation \mathcal{F}_H has same basic Betti, basic Hodge and Hodge numbers. There are many such complex structures, see Example 6.9.

Example 6.2. Consider the product $S^{3,3} = S^3 \times S^3$ of two three dimensional spheres equipped with a complex structure so that there exists a special transverse Kähler structure on a 1-dimensional central foliation \mathcal{F}_H . Then, by our results, $\Omega^*(S^{3,3})$ is quasi-isomorphic to the DGA $\mathcal{A}^* = H_B^*(S^{3,3}) \otimes \bigwedge W$. By $H^1(S^{3,3}) = 0$ and $H^2(S^{3,3}) = 0$, we have $H_B^1(S^{3,3}) = 0$ and the differential $d : W \rightarrow H_B^2(S^{3,3})$ is bijective. Take $W = \langle x, y \rangle$. Then $H_B^2(S^{3,3}) = \langle dx, dy \rangle$. By $\dim H^3(S^{3,3}) = 2$, just two of the elements

$$d(x \wedge dx) = dx \wedge dx, d(y \wedge dy) = dy \wedge dy, d(x \wedge dy) = -d(y \wedge dx) = dx \wedge dy$$

are equal to 0. Take x, y so that $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy \neq 0$. Since the codimension of \mathcal{F}_H is 4, we have $\dim H_B^4(S^{3,3}) = 1$ and thus $H_B^4(S^{3,3}) = \langle dx \wedge dy \rangle$. Thus we have

$$H_B^*(S^{3,3}) = \bigwedge \langle dx, dy \rangle = \langle 1, dx, dy, dx \wedge dy \rangle.$$

Consider the Hodge decomposition

$$H_B^r(S^{3,3}, \mathbb{C}) = \oplus_{p+q=r} H_B^{p,q}(S^{3,3}).$$

Then, by $H_B^{1,1}(S^{3,3}) \neq 0$ and $\dim H_B^2(S^{3,3}) = 2$, we can say that $H_B^{2,0}(S^{3,3}) = H_B^{0,2}(S^{3,3}) = 0$. Thus $H_B^{1,1}(S^{3,3}) = \mathbb{C}\langle dx, dy \rangle$. Take the decomposition $W \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$ with $W^{1,0} = \langle \alpha + \sqrt{-1}\beta \rangle$. Now we have

$$\bar{\partial}(\alpha + \sqrt{-1}\beta) = d\alpha + \sqrt{-1}d\beta.$$

and

$$\bar{\partial}(\alpha - \sqrt{-1}\beta) = 0.$$

By $\langle x, y \rangle = \langle \alpha, \beta \rangle$, we have

$$d\alpha \wedge d\beta \neq 0 \in H_B^4(S^{3,3}, \mathbb{C}) = H_B^{2,2}(S^{3,3}).$$

Hence $\Omega^{*,*}(S^{3,3})$ is quasi-isomorphic to the DBA

$$B^{*,*} = \langle 1, d\alpha, d\beta, \alpha \wedge d\beta \rangle \otimes \bigwedge \langle \alpha + \sqrt{-1}\beta, \alpha - \sqrt{-1}\beta \rangle.$$

We compute

$$H^{1,0}(S^{3,3}) = H^{2,0}(S^{3,3}) = H^{3,0}(S^{3,3}) = H^{0,2}(S^{3,3}) = H^{0,3}(S^{3,3}) = 0$$

and

$$\dim H^{0,1}(S^{3,3}) = \dim H^{2,1}(S^{3,3}) = \dim H^{1,2}(S^{3,3}) = 1.$$

Thus every complex structure on $S^{3,3}$ with a special transverse Kähler structure on a 1-dimensional central foliation \mathcal{F}_H has same basic Betti, basic Hodge and Hodge numbers. Such complex manifolds are constructed as LVM-manifolds associated with complex numbers $(\lambda_1, \dots, \lambda_5)$ with certain conditions (see [17, Section 5]).

Example 6.3. Consider the product $S^{1,3} = S^1 \times S^3$ (resp. $S^{3,3} = S^3 \times S^3$) equipped with a complex structure so that there exists a special transverse Kähler structure on a 1-dimensional central foliation \mathcal{F}_{H_1} (resp. \mathcal{F}_{H_2}). Then the product $S^{1,3} \times S^{1,3}$ has the natural complex structure so that there exists a special transverse Kähler structure on a 2-dimensional central foliation $\mathcal{F}_{H_1 \times H_1}$. The Künneth formula allows us to compute the basic Betti, basic Hodge and Hodge numbers. By Künneth formula we have

$$\dim H_B^i(S^{1,3} \times S^{1,3}) = \begin{cases} 1 & i = 0, 4, \\ 2 & i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim H_B^{p,q}(S^{1,3} \times S^{1,3}) = \begin{cases} 1 & p = q = 0, 2, \\ 2 & p = q = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\dim H^{p,q}(S^{1,3} \times S^{1,3}) = \begin{cases} 1 & (p,q) = (0,0), (4,4), (0,2), (4,2), \\ 2 & (p,q) = (0,1), (4,3), (1,2), (3,2), \\ 4 & (p,q) = (2,2), \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the complex 1-dimensional torus $S^{1,1} = S^1 \times S^1$ and the central foliation $\mathcal{F}_{S^{1,1}}$ on $S^{1,1}$. Then the product $S^{1,1} \times S^{3,3}$ has the natural complex structure so that there exists a special transverse Kähler structure on a 2-dimensional central foliation $\mathcal{F}_{S^{1,1} \times H_2}$. By Künneth formula we have

$$\begin{aligned} \dim H_B^i(S^{1,1} \times S^{3,3}) &= \begin{cases} 1 & i = 0, 4, \\ 2 & i = 2, \\ 0 & \text{otherwise} \end{cases} \\ &= \dim H_B^i(S^{1,3} \times S^{1,3}), \\ \dim H_B^{p,q}(S^{1,1} \times S^{3,3}) &= \begin{cases} 1 & p = q = 0, 2, \\ 2 & p = q = 1, \\ 0 & \text{otherwise} \end{cases} \\ &= \dim H_B^{p,q}(S^{1,3} \times S^{1,3}) \end{aligned}$$

but

$$\dim H^{p,q}(S^{1,1} \times S^{3,3}) \neq \dim H^{p,q}(S^{1,3} \times S^{1,3})$$

for some p, q . Indeed, $\dim H^{1,0}(S^{1,1} \times S^{3,3}) = 1$ but $\dim H^{1,0}(S^{1,3} \times S^{1,3}) = 0$. Thus, in general, the Hodge numbers depend on a complex structure.

6.2. Nilmanifolds. Let N be a simply connected nilpotent Lie group. We suppose that N admits a lattice Γ i.e. cocompact discrete subgroup. A compact homogeneous space $\Gamma \backslash N$ is called a *nilmanifold*. It is known that a nilmanifold admits a Kähler structure if and only if it is a torus (see [2, 9]).

Denote by \mathfrak{n} the Lie algebra of N . Let J be an endomorphism of \mathfrak{n} satisfying $J \circ J = -\text{id}$ and $[JA, JB] = [A, B]$ for any $A, B \in \mathfrak{n}$. Then J induces a complex structure on $\Gamma \backslash N$. Such complex structure is called *abelian*. We assume that \mathfrak{n} is non-abelian and 2-step i.e. $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$. Let C be the center of N and $\psi : N \rightarrow N/C$ the quotient map. Then we have the holomorphic principal torus bundle

$$T \hookrightarrow \Gamma \backslash N \rightarrow M$$

where T and M are complex tori $\Gamma \cap C \backslash C$ and $M = \psi(\Gamma \backslash N)$ respectively. Let \mathfrak{c} be the sub-algebra of \mathfrak{n} corresponding to C . Consider the complex $\bigwedge^* \mathfrak{n}^*$ of left- N -invariant differential forms. Take $W \subset \bigwedge^1 \mathfrak{n}^*$ which is dual to \mathfrak{c} . Then we have $dW \subset \Omega^{1,1}(\Gamma \backslash N)$. Thus, in this case, $\Gamma \backslash N$ admits a special transverse Kähler structure on \mathcal{F}_C .

We study the properties of nilmanifolds admitting special transverse Kähler structures on central foliations

Proposition 6.4. *Let $\Gamma \backslash N$ be a nilmanifold with a (not necessarily left-invariant) complex structure J . We assume that M admits a transverse Kähler structure on a k -dimensional central foliation \mathcal{F}_H . Suppose that \mathcal{F}_H is regular i.e. H is a compact and the H -action is free. Then $\Gamma \backslash N$ is biholomorphic to a holomorphic principal*

torus bundle over a complex torus. In particular, $\Gamma \backslash N$ is 2-step nilmanifold (see [21]).

Proof. By the assumption, $\Gamma \backslash N$ admits a holomorphic principal torus H bundle structure $\Gamma \backslash N \rightarrow B$ so that the base space is a compact Kähler manifold. Since $\Gamma \backslash N$ is an aspherical manifold with $\pi_1(\Gamma \backslash N) \cong \Gamma$, we can say that B is a compact aspherical manifold so that $\pi_1(B)$ is a finitely generated nilpotent group. By results in [9], [2] and [1], B is a complex torus. Thus $\Gamma \backslash N$ is a holomorphic principal torus bundle over a complex torus. \square

We are interested in the non-regular case.

Proposition 6.5. *Let $\Gamma \backslash N$ be a nilmanifold with a (not necessarily left-invariant) complex structure J . We assume that $\Gamma \backslash N$ admits a special transverse Kähler structure on a central foliation \mathcal{F}_H . If H is complex 1-dimensional, then $\Gamma \backslash N$ is diffeomorphic to a 2-step nilmanifold.*

Proof. Let M be a compact complex n -dimensional manifold which admits a special transverse Kähler structure on a k -dimensional central foliation \mathcal{F}_H . Then we have an isomorphism

$$H^{2n}(M, \mathbb{C}) \cong H_B^{n-k, n-k}(M) \otimes \bigwedge^{2k} W_{\mathbb{C}}.$$

Hence, for the mixed Hodge structure as in Theorem 5.10, $H^{2n}(M, \mathbb{C})$ is generated by elements of bi-degree $(n+k, n+k)$.

Consider nilmanifold $\Gamma \backslash N$. Then the DGA $\bigwedge \mathfrak{n}^*$ is the minimal model of $\Omega^*(\Gamma \backslash N)$ (see [9]). If $\Gamma \backslash N$ admits a special transverse Kähler structure on a central foliation \mathcal{F}_H , then by Theorem 5.10, the minimal model $\bigwedge \mathfrak{n}_{\mathbb{C}}^*$ of $\Omega^*(\Gamma \backslash N)$ admits a bigrading $\bigwedge \mathfrak{n}_{\mathbb{C}}^* = \bigoplus \mathcal{M}_{p,q}^*$. Denote $\mathcal{M}_w^* = \bigoplus_{p+q=w} \mathcal{M}_{p,q}^*$ and $m_w = \dim \mathcal{M}_w^1$. Since $\dim \mathcal{M}^1 = \dim \mathfrak{n}_{\mathbb{C}}^* = 2n$, we have $\sum_{w \geq 1} m_w = 2n$. Since we have $H^{2n}(M, \mathbb{C}) = \bigwedge^{2n} \mathfrak{n}_{\mathbb{C}}^* = \bigwedge^{2n} \bigoplus_w \mathcal{M}_w^1$, we have $\sum_{w \geq 1} w m_w = 2n + 2k$. Let $k = 1$. Then $\sum_{w \geq 2} (w-1)m_w = 2$ and hence we have $m_2 = 2$ and $m_i = 0$ for $i \leq 3$, or $m_2 = 0$, $m_3 = 1$ and $m_i = 0$ for $i \leq 4$. We can say $d\mathcal{M}_1 = 0$ and $\bigwedge \mathfrak{n}_{\mathbb{C}}^* = \bigwedge \mathcal{M}_1^1 \otimes \bigwedge V$ with $dV \subset \bigwedge^2 \mathcal{M}_1^1$. This implies that \mathfrak{n} is 2-step. \square

We suggest the following problem.

Problem 6.6. For $s \geq 3$ and $k \geq 2$, does there exist a s -step nilmanifold admitting a special transverse Kähler structure on a k -dimensional non-regular central foliation \mathcal{F}_H ?

6.3. Vaisman manifolds. Let (M, J) be a compact complex manifold with a Hermitian metric g . We consider the fundamental form $\omega = g(-, J-)$ of g . The metric g is locally conformal Kähler (LCK) if we have a closed 1-form θ (called the Lee form) such that $d\omega = \theta \wedge \omega$. It is known that if $\theta \neq 0$ and θ is non-exact, then (M, J) does not admit a Kähler structure. Let ∇ be the Levi-Civita connection of g . A LCK metric g is a Vaisman metric if $\nabla \theta = 0$.

If g is Vaisman, then we have the following results ([25], [26]):

- Take the dual vector fields A and B corresponding to the 1-forms θ and $-\theta \circ J$ respectively, we have $A = JB$, $L_A J = 0$, $L_B J = 0$, $L_A g = 0$, $L_B g = 0$ and $[A, B] = 0$.

- By these conditions, the holomorphic vector field $B - \sqrt{-1}A$ gives a holomorphic foliation \mathcal{F} .
- The basic form $d(\theta \circ J)$ is a transverse Kähler structure.
- We denote by $\text{Aut}_0(M, g)$ the identity component of the group of holomorphic isometries, by \mathfrak{h} the abelian sub-algebra $\langle A, B \rangle$ of the Lie algebra of $\text{Aut}_0(M, g)$ and by H the connected Lie subgroup of $\text{Aut}_0(M, g)$ which corresponds to \mathfrak{h} . Let T be the closure of H in $\text{Aut}_0(M, g)$. Then T is a torus.

Thus a compact Vaisman manifold M admits a special transverse Kähler structure on the 1-dimensional central foliation \mathcal{F}_H . Hence, taking $W = \langle \theta, \theta \circ J \rangle$ our results can be applied to a compact Vaisman manifold. The cohomology of the DGA

$$\mathcal{A}^* = H_B^*(M) \otimes \bigwedge \langle \theta, \theta \circ J \rangle$$

is isomorphic to the de Rham cohomology of M and the cohomology of DBA

$$\mathcal{B}^{*,*} = H_B^{*,*}(M) \otimes \bigwedge \langle \theta + \sqrt{-1}\theta \circ J, \theta - \sqrt{-1}\theta \circ J \rangle$$

is isomorphic to the Dolbeault cohomology of M . We can easily compute

$$H^1(M, \mathbb{C}) = H_B^1(M) \oplus \langle \theta \rangle = H_B^{1,0}(M) \oplus H_B^{0,1}(M) \oplus \langle \theta \rangle.$$

This implies well known fact that the first Betti number of a compact Vaisman manifold is odd (see [25]). We have the mixed Hodge structure

$$H^1(M, \mathbb{C}) = H_{1,0}^1 \oplus H_{0,1}^1 \oplus H_{1,1}^1$$

with $\dim H_{1,1}^1 = 1$ as in Theorem 5.10. We notice that Vaisman metrics are closely related to Sasakian structures. We can also obtain nice de Rham models of Sasakian manifolds like the above DGA (see [24]) and we can develop Morgan's mixed Hodge theory on Sasakian manifolds (see [13]).

Since we have $\bar{\partial}(\theta + \sqrt{-1}\theta \circ J) = \sqrt{-1}d(\theta \circ J)$ and $\bar{\partial}(\theta - \sqrt{-1}\theta \circ J) = 0$, we can easily obtain an isomorphism of DGA

$$\mathcal{A}^* \otimes \mathbb{C} \cong \text{Tot}^* \mathcal{B}^{*,*}.$$

Hence, by Theorem 4.12, we can say the following (cf.[25, Theorem 3.5]).

Corollary 6.7. *Let M be a compact complex manifold. We suppose that M admits a Vaisman metric. Then the two DGAs $(\Omega^*(M) \otimes \mathbb{C}, d)$ and $(\Omega^*(M) \otimes \mathbb{C}, \bar{\partial})$ are quasi-isomorphic. Hence we have an isomorphism between the complex valued de Rham cohomology and the Dolbeault cohomology.*

Remark 6.8. On compact Kähler manifold M , by the $\partial\bar{\partial}$ -lemma, we can say that two DGAs $(\Omega^*(M) \otimes \mathbb{C}, d)$ and $(\Omega^*(M) \otimes \mathbb{C}, \bar{\partial})$ are quasi-isomorphic (see [19]).

Example 6.9. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be complex numbers so that $0 < |\lambda_n| \leq \dots \leq |\lambda_1| < 1$. A *primary Hopf manifold* M_Λ is the quotient of $\mathbb{C}^n - \{0\}$ by the group generated by the transformation $(z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n)$. It is known that any M_Λ admits a Vaisman metric (see[12]). For any Λ , M_Λ is diffeomorphic to $S^{1,2n-1} = S^1 \times S^{2n-1}$. On the other hand, the complex structure on M_Λ varies. If $\lambda_n = \dots = \lambda_1$, then M_Λ is a holomorphic principal torus bundle over $\mathbb{C}P^{n-1}$. But, in general, a holomorphic principal torus bundle structure does not exist on M_Λ . By Example 6.1 and the above arguments, we can obtain explicit representatives of de Rham, Dolbeault, basic de Rham and Basic Dolbeault cohomologies of M_Λ by using a Vaisman metric on M_Λ .

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